

# 1 HW 1

## 1.1 Exercise 1

In the notes we showed how to prove the following:

$$\mathbb{E}[X] = \int_0^\infty P(X > t) dt$$

This follows from an application of Fubini's theorem applied to the integral:

$$\begin{aligned} \mathbb{E}[X] &= \int_0^\infty u f_X(u) du \\ &= \int_0^\infty \int_0^\infty \mathbb{1}(0 \leq t \leq u) dt f_X(u) du \\ &= \int_0^\infty \int_0^\infty \mathbb{1}(0 \leq t \leq u) f_X(u) du dt \\ &= \int_0^\infty P(X > t) dt \end{aligned}$$

Use Fubini's theorem to show the following equality:

$$\int_x^\infty (u - x) f_X(u) du = \int_x^\infty S_X(u) du.$$

Use indicator variables.

## 1.2 Exercise 2

In the notes we showed that when  $\mathbb{E}[|X|] < \infty$  that

$$\lim_{u \rightarrow \infty} uP(X > u) = 0. \tag{1}$$

Use Markov's inequality:

$$P(X > a) \leq \frac{\mathbb{E}[X]}{a}$$

to show that when  $\mathbb{E}[X^2] < \infty$   $\lim_{u \rightarrow \infty} uP(X > u) = 0$ .

## 1.3 Exercise 3

Suppose we have a survival time,  $X$  that is exponentially distributed with rate  $\theta$ .

$$X \mid \Theta = \theta \sim \text{Exp}(\theta).$$

Further, suppose that  $\Theta$  is randomly distributed with a Gamma distribution with parameters  $\alpha, \beta$ :

$$f_\Theta(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\beta\theta) \mathbb{1}(\theta > 0).$$

1. What is the survival function of the marginal random variable  $X$ ? (Hint: calculate  $\mathbb{E}_\Theta [e^{-\Theta t}]$ )
2. What is the hazard function for  $X$  marginal over  $\Theta$ ?
3. Use R and your favorite plotting package (base R, ggplot2, etc.) to plot the hazard function for several values of  $\alpha, \beta$ .

## 1.4 Exercise 4

Let  $X_i \stackrel{\text{iid}}{\sim} \text{Exp}(\beta)$  for  $i \in \{1, \dots, n\}$  and let  $Y = \sum_{i=1}^n X_i$ .

1. Show that  $Y \sim \text{Gamma}(n, \beta)$ .
2. Show that the survival function for  $Y$  can be expressed in terms of the upper incomplete gamma function.
3. Express the mean residual lifetime in terms of the upper incomplete gamma function

As a reminder, the upper incomplete gamma function,  $\Gamma(\alpha, x)$  is defined as:

$$\int_x^\infty u^{\alpha-1} e^{-u} du.$$

Note that the order of the arguments differs by source. Some textbooks define it as  $\Gamma(x, \alpha)$ , whereas Abramowitz and Stegun 1948 defines it as I have above. Either is fine as long as you are consistent in your definition.

## 1.5 Exercise 5

The mean residual lifetime, or  $\mathbb{E}[X - x | X > x]$  of a random variable  $X$  is  $x + 10$ . What is the hazard function and the survival function of  $X$ ? Hint: Use the definition of MRL in terms of the survival function,

$$\frac{\int_x^\infty S_X(t) dt}{S_X(x)} = x + 10$$

and try to manipulate this expression to get an expression relating the density to the survival function of  $X$ .

## References

- [1] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. Vol. 55. US Government printing office, 1948.